The Aligarh Bulletin of Mathematics
Department of Mathematics
Aligarh Muslim University
ALIGARH-202002 (India)
Phone: 0571-701019

Dr. Saleh A. Al-Mezel
Deptt. of Maths.
King Abdul Aziz Univ.
P.O. Box 80203
Jeddah-21589 - Saudi Arabia

Ref: ABM/ July 16/07

Dear Dr. Al-Mezel

I am glad to inform that your paper entitled A result on coincidence points (Co-auth: A. E. Al-Mazrooei)

has been accepted for publication in *The Aligarh Bulletin of Mathematics* and likely to appear in Vol. 26 No. – year 2007

You are requested to send a demand draft for ~~Rs~~/US $ 40=00 as the publication charges. The draft may be made in the name of

The Finance Officer
Aligarh Muslim University, Aligarh

and be sent to me at your earliest. An invoice of the same is being attached herewith.

Thank you very much for your interest in our bulletin and with best regards.

Yours sincerely
Zafar Ahsan
(Dr. ZAFAR AHSAN)
Managing Editor
The Aligarh Bulletin of Mathematics
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 (INDIA)

YAHOO! MAIL

Print - Close Window

Date: 9 Jun 2007 10:10:21 -0000

From: "zafar ahsan" <zafar.ahsan@rediffmail.com>

To: "Dr.Saleh almezel" <mathsaleh@yahoo.com>

Subject: Re: paper

Dear Dr. Saleh Al-Mazel

I am glad to inform you that your paper entitled " "A Result On Coincidence Points" has got positive reports from our referee and thus has been accepted for publication in The Aligarh bulletin of Mathematics. It is likely to appear in Vol. 27 (2008).

A formal acceptance letter and the bill for publication charges are being despatched by seperate post.

Thanking you for your interest in our bulletin and with best regards.

Sincerely yours

Prof. Zafar Ahsan

Managing Editor
The Aligarh Bulletin of Mathematics
Department of Mathematics
Aligarh Muslim University
Aligarh - 202 002(India).

A Result On Coincidence Points

Saleh A. Al-Mezel And Abdullah E. Al-Mazrooei

Department of Mathematics
King Abdul Aziz University
P.O.Box 80203, Jeddah-21589
Saudi Arabia

Abstract: In this paper we prove a coincidence point result in the setting of metric spaces under some general contractive condition. Consequently, we improve and generalize various known results existing in the literature.

1. Introduction and Preliminaries

Using the concept of Hausdorff metric, many authors have proved fixed point and coincidence point results in the setting of metric spaces. For example, using the Hausdorff metric, Nadler [14] has introduced a notion of multivalued contraction maps and proved a multivalued version of the Banach contraction principle which states that each multivalued contraction map on a complete metric space with values as closed bounded subsets of. the space, has a fixed point. Since then various fixed point results concerning multivalued contractions have been appeared. For example, see [1-3,5,7,13,15]. In [8], Kaneko has generalized a notion of multivalued contraction maps by introducing a notion of multivalued f-contractions and proved coincidence point result for such maps with commutativity condition, extending the corresponding results of Jungck [6], Nadler [14] and others. This result has been generalized in different directions. For example, see [10,12,16]. Among others, Latif and Beg [11] have proved a coincidence point result for non-commuting maps, which is an improved version of the result of Kaneko [8].

In this paper we prove a coincidence point result under some general contractive condition, which generalizes the corresponding results of Latif and Beg [11], Daffer and Kaneko [3], and many others.

Throughout this paper, (X,d) is a metric space and $CB(X)$ is the family of nonempty closed bounded subsets of X. For any $A, B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{x\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is a distance from the point a to the subset B. It is well known that H is a metric on $CB(X)$ and is known as the Hausdorff metric on $CB(X)$.

We also use the following notions.

Let $T : X \to CB(X)$ be a multivalued map and $f : X \to X$ a single-valued map.

i) T is called *contraction* [14] if there exists a constant $h \in (0,1)$ such that

$$H(T(x),T(y)) \le hd(x,y), \qquad x, y \in X.$$

ii) T is called *f-contraction* [8] if there exists a constant $h \in (0,1)$ such that

$$H(T(x),T(y)) \le h\, d(f(x), f(y)), \qquad x, y \in X.$$

iii) T is called generalized f-contraction [16], if there exists a constant $h \in (0,1)$ such that for all $x, y \in X$

$$H(Tx,Ty) \le h \max\left\{d(fx, fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty) + d(fy,Tx)]\right\}.$$

In particular, if $f = I$, the identity map on X then each multivalued f-contraction map is a contraction, each multivalued generalized f-contraction map is a general contraction defined by Daffer and Kaneko [3]. A point $x \in X$ is called a *fixed point* of T if $x \in T(x)$ and the set of fixed points of T is denoted by $Fix(T)$. A point $x \in X$ is called a *coincidence point* of f and T if $f(x) \in T(x)$. We denote by $C(f \cap T)$ the set of coincidence points of f and T.

A real valued function f on X is called lower semi-continuous if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \le \underline{\lim}_{n\to\infty} f(x_n)$.

Kaneko and Sessa [9] have obtained the following coincidence point result for compatible maps.

Theorem 1.1 *Let (X,d) be a complete metric space, $f : X \to X$ and $T : X \to CB(X)$, be a multivalued generalized f-contraction compatible continuous maps such that $T(X) \subseteq f(X)$. Then, there exists a point $x_0 \in X$ such that $f(x_0) \in T(x_0)$.*

2. A Result

We prove a coincidence point result for non-compatible maps.

Theorem 2.1 *Let (X,d) be a metric space, and let f be a continuous self map of X with $f(X)$ complete. Let $T: X \to CB(X)$ be a generalized f-contraction map such that $T(X) \subset f(X)$. Then there exists $x_0 \in X$ such that $f(x_0) \in T(x_0)$, provided the map $y \to d(y,Ty)$ is lower semicontinuous.*

Proof. Suppose $f(t) \notin T(t)$, for all $t \in X$. Since $T(X) \subset f(X)$ so for any $x \in X$ there exists a $y \in X$ such that $f(y) \in T(x)$. If $h = 0$, then we have

$H(T(x),T(y)) = 0,$

thus, $T(x) = T(y)$, which means that $f(y) \in T(y)$. Now, assume that $h \neq 0$, and choose a number c with $1 < c < \frac{1}{h}$. Since $d(f(x),T(x)) = \inf\{d(f(x),z) \mid z \in T(x)\}$, and $f(y) \in T(x)$ then $d((fx),T(x)) \leq d(f(x),f(y))$. Thus

$0 < d(f(x),f(y)) \leq cd(f(x),T(x))$, because $c > 1$ and $f(x) \neq f(y)$. Now, by the definition of the Hausdorrf metric we have

$$d(f(y),T(y)) \leq H(T(x),T(y))$$

$$\leq h\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty)+d(fy,Tx)]\right\}. \tag{1.1}$$

Since $f(y) \in T(x)$ then $d(f(y),T(x)) = 0$. Also, if $d(f(y),T(y))$ is the maximum, then we get

$$d(f(y),T(y)) \leq hd(f(y),T(y)),$$

which is not possible because $0 < h < 1$. Thus (1.1) becomes

$$d(f(y),T(y)) \leq H(T(x),T(y))$$

$$\leq h\max\left\{d(f(x),f(y)), d(f(x),T(x)), \frac{1}{2}d(f(x),T(y))\right\}.$$

We need to examine the following three cases. First suppose that

$$d(f(x),f(y)) = \max\left\{d(f(x),f(y)), d(f(x),T(x)), \frac{1}{2}d(f(x),T(y))\right\}.$$

Then

$$\begin{aligned} d(f(y),T(y)) &\le hd(f(x),f(y)) \\ &\le hcd(f(x),T(x)) \\ &< d(f(x),T(x)), \end{aligned}$$

because $ch<1$. Also, since $d(f(y),(y))\le hd(f(x),f(y))$ then

$$-d(f(y),T(y))\ge -hd(f(x),f(y))$$

so

$$\begin{aligned} d(f(x),T(x))-d(f(y),T(y)) &\ge d(f(x),T(x))-hd(f(x),f(y)) \\ &\ge \frac{1}{c}d(f(x),f(y))-hd(f(x),f(y)) \\ &\ge (\frac{1}{c}-h)d(f(x),f(y)). \end{aligned}$$

Second, suppose that

$$d(f(x),T(x))=\max\left\{d(f(x),f(y)),d(f(x),T(x)),\frac{1}{2}d(f(x),T(y))\right\}.$$

Then,

$$\begin{aligned} d(f(y),T(y)) &\le hd(f(x),T(x)) \\ &< d(f(x),T(x)), \end{aligned}$$

because $h<1$. Also since

$$d(f(y),T(y))\le hd(f(x),T(x))$$

then,

$$-d(f(y),T(y))\ge -hd(f(x),T(x)),$$

thus,

$$\begin{aligned} d(f(x),T(x))-d(f(y),T(y)) &\ge d(f(x),T(x))-hd(f(x),T(x)) \\ &\ge (1-h)d(f(x),T(x)) \\ &\ge (\frac{1-h}{c})d(f(x),f(y)) \\ &= (\frac{1}{c}-\frac{h}{c})d(f(x),f(y)) \\ &\ge (\frac{1}{c}-h)d(f(x),f(y)), \end{aligned}$$

because $\frac{h}{c}\le h$. Third, suppose that,

$$\frac{1}{2}d(f(x),T(y))=\max\left\{d(f(x),f(y)),d(f(x),T(x)),\frac{1}{2}d(f(x),T(y))\right\}.$$

Then,

$$d(f(y),T(y)) \le \frac{h}{2} d(f(x),T(y)). \tag{1.2}$$

Since,

$$d(f(x),T(y)) \le d(f(x),f(y)) + d(f(y),T(y))$$

since $\frac{h}{2} > 0$, then we have

$$\frac{h}{2} d(f(x),T(y)) \le \frac{h}{2}[d(f(x),f(y)) + d(f(y),T(y))]$$

So (1.2) becomes,

$$d(f(y),T(y)) \le \frac{h}{2}[d(f(x),f(y)) + d(f(y),T(y))],$$

thus,

$$(1-\frac{h}{2}) d(f(y),T(y)) \le \frac{h}{2} d(f(x),f(y)).$$

Since $\frac{h}{2} < 1$, we have

$$d(f(y),T(y)) < (\frac{h}{2})(\frac{2}{2-h}) d(f(x),f(y)),$$

and thus,

$$\begin{aligned} d(f(y),T(y)) &\le (\frac{h}{2-h}) d(f(x),f(y)) \\ &\le (\frac{hc}{2-h}) d(f(x),T(x)) \end{aligned}$$

since $hc < 1$ and $2-h > 1$ then $\frac{hc}{2-h} < 1$, thus

$$d(f(y),T(y)) < d(f(x),T(x)).$$

Also, since

$$d(f(y),T(y)) \le (\frac{h}{2-h}) d(f(x),f(y))$$

thus

$$-d(f(y),T(y)) \ge -(\frac{h}{2-h}) d(f(x),f(y))$$

hence,

$$\begin{aligned} d(f(x),T(x))-d(f(y),T(y)) &\geq d(f(x),T(x))-(\frac{h}{2-h})d(f(x),f(y)) \\ &\geq \frac{1}{c}d(f(x),f(y))-(\frac{h}{2-h})d(f(x),f(y)) \\ &\geq (\frac{1}{c}-\frac{h}{2-h})d(f(x),f(y)) \\ &\geq (\frac{1}{c}-h)d(f(x),f(y)), \end{aligned}$$

because $\frac{h}{2-h} \leq h$. Thus for all three cases we have

$$\text{(i)} \quad d(f(y),T(y)) < d(f(x),T(x))$$

and

$$\text{(ii)} \quad d(f(x),T(x))-d(f(y),T(y)) \geq (\frac{1}{c}-h)d(f(x),f(y)).$$

Now, by (i) we conclude that

$$\inf_{t \in X} d(f(t),T(t)) = 0.$$

Define a map $\psi : f(X) \to \Re$ by

$$\psi(f(t)) = (\frac{1}{c}-h)^{-1} d(f(t),T(t)) \qquad \text{for every } t \in X.$$

Then ψ is lower semicontinuous function. By using (ii) we get

$$\begin{aligned} d(f(x),f(y)) &\leq (\frac{1}{c}-h)^{-1} d(f(x),T(x)) - (\frac{1}{c}-h)^{1} d(f(y),T(y)) \\ &\leq \psi(f(x)) - \psi(f(y)). \end{aligned}$$

Thus,

$$\psi(f(y)) + d(f(x),f(y)) \leq \psi(f(x)).$$

By [4], we conclude that there exists an element $y_0 \in X$ such that

$$\psi(f(y_0)) = \inf_{t \in X} \psi(f(t)) = 0.$$

Thus,

$$\psi(f(y_0)) = (\frac{1}{c}-h)^{-1} d(f(y_0),T(y_0)) = 0.$$

Since $(\frac{1}{c}-h)^{-1} \neq 0$, then $d(f(y_0),T(y_0)) = 0$. Since T is closed, thus $f(y_0) \in T(y_0)$. This contradicts our assumption that $f(t) \notin T(t)$, for all $t \in X$. Therefore, there exists $x_0 \in X$ such that $f(x_0) \in T(x_0)$.

Remark 2.2 1) Theorem 2.1 generalizes a corresponding result due to Latif and Beg [11] and contains a fixed point result of Daffer and Kaneko [3] as a special case. 2) It is observed that Theorem 2.1 is can be obtained from Theorem 2 in [16]. However, our proof is simple and completely different than as given in [16].

References

[1] J. S. Bae, "Fixed point theorem for weakly contractive multivalued maps", *J. Math. Anal. Appl.,* 284 (2003), 690-697.

[2] H. Covitz, S. B. Nadler, "Multivalued contraction mappings in generalized metric space", *Israel J. Math.* 8 (1970) 5-11.

[3] P. Z. Daffer and H. Kaneko, "Fixed points generalized contractive multi-valued mappings", *J. Math. Anal. Appl.,* 192 (1995), 655-666.

[4] I. Ekland, "Nonconvex minimization problems", *Bull. Amer. Math. Soc.*, 1 (1979), 443-474.

[5] Y. Feng and S. Liu, " Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings", *J. Math. Anal. Appl.,* 317 (2006) 103-112.

[6] G. Jungck, " Commuting maps and fixed points", *Amer. Math. Monthly,*83 (1976), 261-263.

[7] G. Jungck and B.E Rhoades, "Fixed points for set valued functions without continuity", *Indian J. Pure Appl. Math.,* 29(3) (1998), 227-238.

[8] H. Kaneko, "Single-valued and multivalued *f*-contractions" , *Boll. Un. Mat. Ital.,*6 (1985),29-33.

[9] H. Kaneko and S. Sessa, "Fixed point theorem for compatible multivalued and single-valued mappings", Int. J. math. Math. Sci., 12 (1989), 257-262.

[10] A. Latif and Alaa Abu-Hajar, "On coincidence and hybrid fixed point", *Radovi Matematicki,* 11 (2002), 119-123.

[11] A. Latif and I. Beg, "Geometric Fixed Pointsfor Single and Multivalued Mappings" *Demon. Math.,* Vol. XXX No. 4 (1997), 791-795.

[12] A. Latif and Tweddle, "Some results on coincidence points", *Bull. Austral. Math. Soc.,* 59 (1999), 111-117.

[13] N. Mizoguchi and W. Takahashi, "Fixed points theorems for multivalued mappings on complete metric spaces", *J. Math. Anal. Appl.*, 141 (1989), 177-188.

[14] S. B. Nadler, "Multivalued contraction mappings", *Pacific J. Math.*, 30(1969) 475-488.

[15] S. V. R. Naidu, " Fixed point theorems for a broad class of multimaps", *Nonlinear Anal.*, 52 (2003) 961-969.

[16] H. K. Pathak and M. S. Khan, "Fixed and coincidence points of hybrid mappings", *Archivum Mathematicum(BRNO)*, 38 (2002), 201-208.